Trinity College

Semester One Examination, 2017

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 3

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	11	11	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

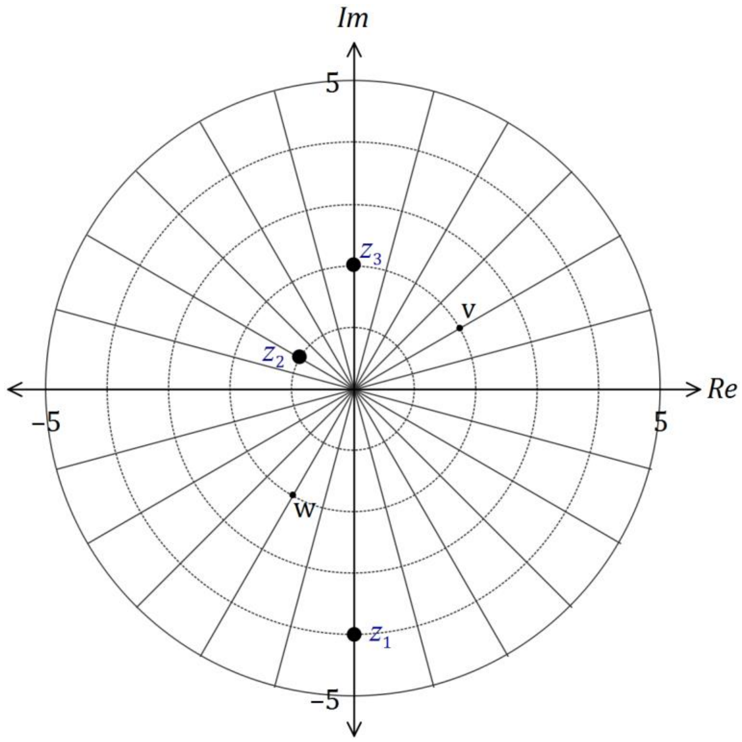
Section Two: Calculator-assumed **65% (98 Marks)**

This section has **eleven (11)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(6 marks)**

The complex numbers v and w are shown on the Argand diagram below.



On the diagram, clearly mark the complex numbers

(a) $z_1 = vw$. (2 marks)

Solution
Multiply moduli and add arguments
Specific behaviours
✓ correct modulus ✓ correct argument

(b) $z_2 = \dfrac{v}{w}$. (2 marks)

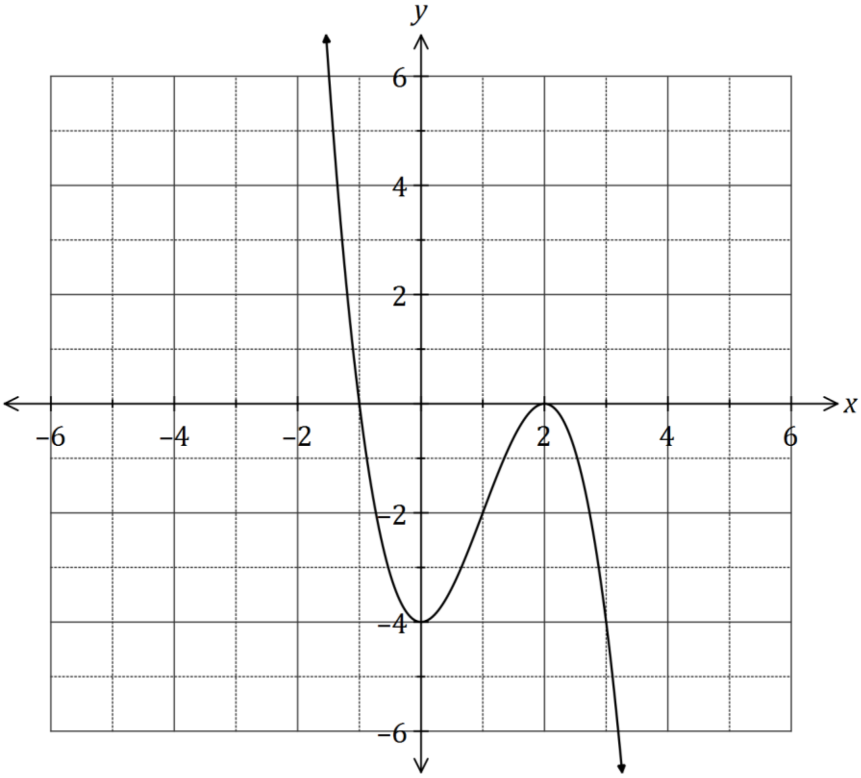
Solution
Divide moduli and subtract arguments
Specific behaviours
✓ correct modulus ✓ correct argument

(c) $z_3 = v - iw$. (2 marks)

Solution
Rotate w 90° anticlockwise and then treat as vector addition
Specific behaviours
✓ correct modulus ✓ correct argument

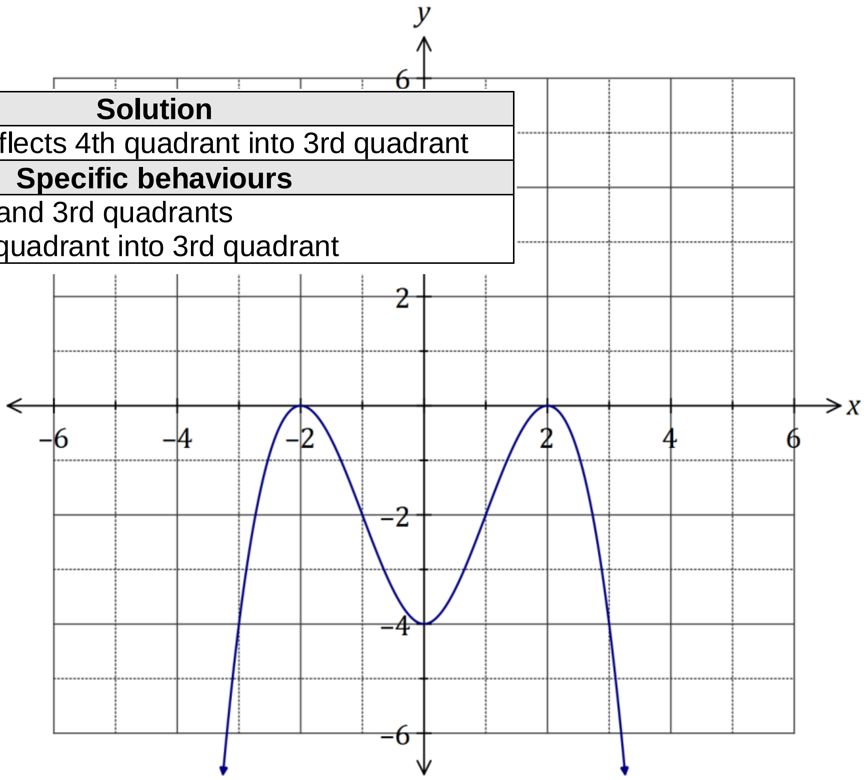
Question 10 **(8 marks)**

The graph of $y = f(x)$ is drawn below.



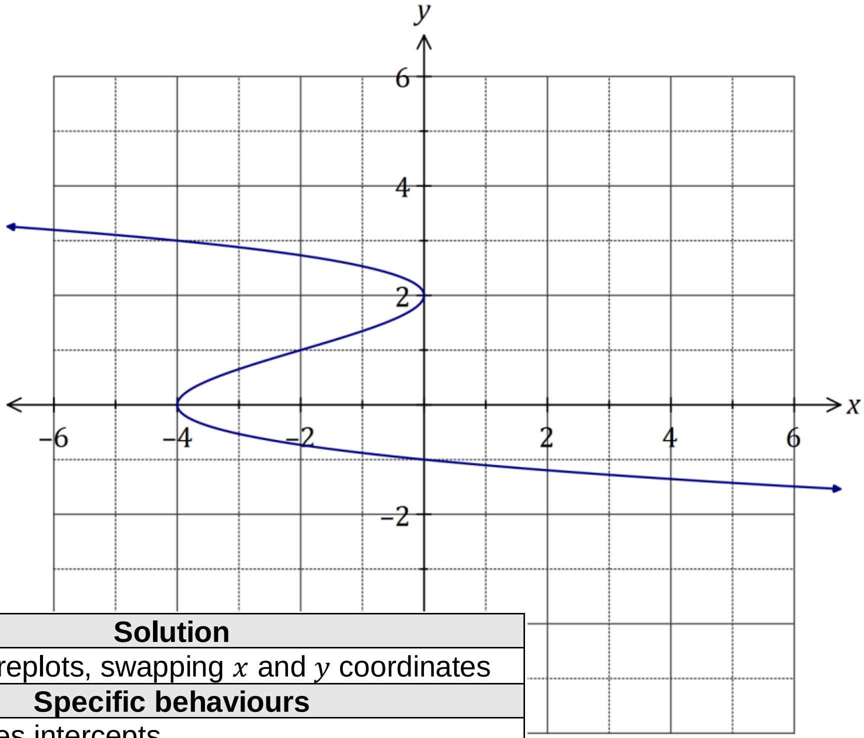
On the axes provided, sketch the graphs of

(a) $y = f(|x|)$. (2 marks)



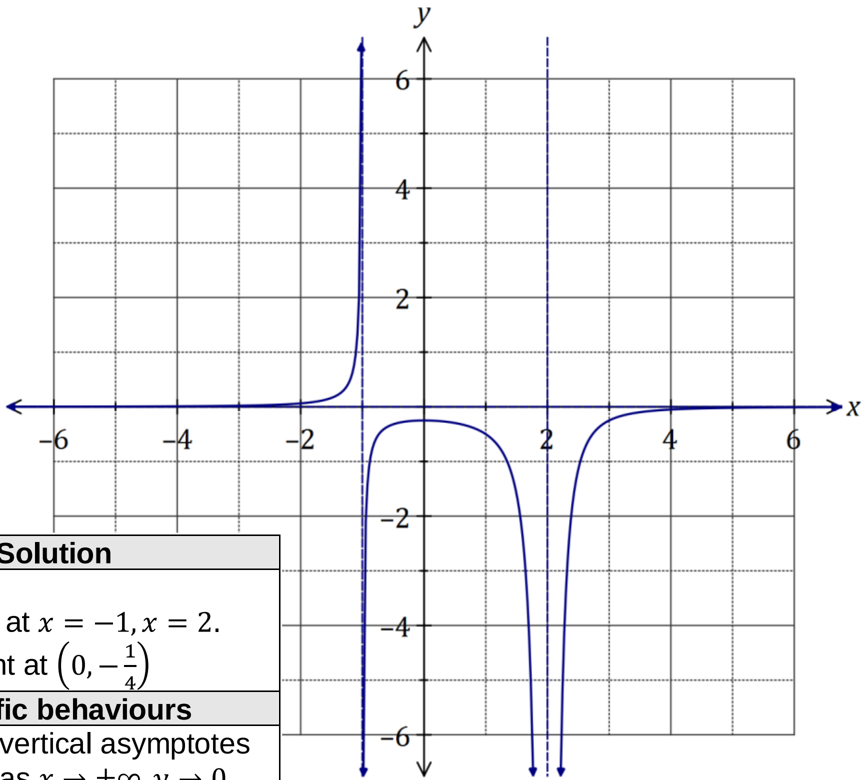
Solution
See graph - reflects 4th quadrant into 3rd quadrant
Specific behaviours
✓ erases 2nd and 3rd quadrants ✓ reflects 4th quadrant into 3rd quadrant

(b) $y = f^{-1}(x)$. (2 marks)



Solution
See graph - replots, swapping x and y coordinates
Specific behaviours
✓ correct axes intercepts ✓ correct graph

(c) $y = \dfrac{1}{f(x)}$. (4 marks)



Solution
See graph. Asymptotes at $x = -1, x = 2$. Turning point at $\left(0, -\frac{1}{4}\right)$
Specific behaviours
✓ indicates vertical asymptotes ✓ indicates as $x \to \pm\infty, y \to 0$ ✓ indicates turning point ✓ correct graph

Question 11 **(9 marks)**

The position vectors of particles A and B are $\mathbf{r_A} = \begin{pmatrix} 15 - t \\ 4 - 3t \end{pmatrix}$ and $\mathbf{r_B} = \begin{pmatrix} 3t - 8 \\ 5 - t^2 \end{pmatrix}$, where t is the time in seconds, $t \geq 0$, and distances are measured in metres.

(a) Determine the speed of particle B when $t = 2$. (2 marks)

Solution
$\mathbf{v_B} = \begin{pmatrix} 3 \\ -2t \end{pmatrix}$, $\lvert v_B(2) \rvert = \left\lvert \begin{pmatrix} 3 \\ -4 \end{pmatrix} \right\rvert = 5 \text{ m/s}$
Specific behaviours
✓ differentiates position vector ✓ determines speed

(b) Determine the Cartesian equation for the path of particle A. (3 marks)

Solution
$x = 15 - t \Rightarrow t = 15 - x$ $y = 4 - 3t = 4 - 3(15 - x)$ $y = 3x - 41, x \leq 15$
Specific behaviours
✓ expresses t in terms of x ✓ substitutes ✓ includes domain restriction

(c) Determine where the paths of the particles cross and whether the particles meet. Justify your answer. (4 marks)

Solution
Let $\mathbf{r_A} = \begin{pmatrix} 15 - s \\ 4 - 3s \end{pmatrix}$ where s is time. Then $15 - s = 3t - 8$ and $4 - 3s = 5 - t^2$. Solving simultaneously (CAS): $s = 8, t = 5$ or ~~$s = 65, t = -14$~~ $\mathbf{r} = \begin{pmatrix} 7 \\ -20 \end{pmatrix}$ Hence paths cross at $\begin{pmatrix} 7 \\ -20 \end{pmatrix}$ but particles do not meet, as times that they are at this location are different.
Specific behaviours
✓ equates coefficients ✓ solves equations ✓ determines location ✓ interprets solution

Question 12 **(9 marks)**

A function is defined by $f(x) = \frac{x^2+4x-12}{3x-7}$, $x \neq 0$.

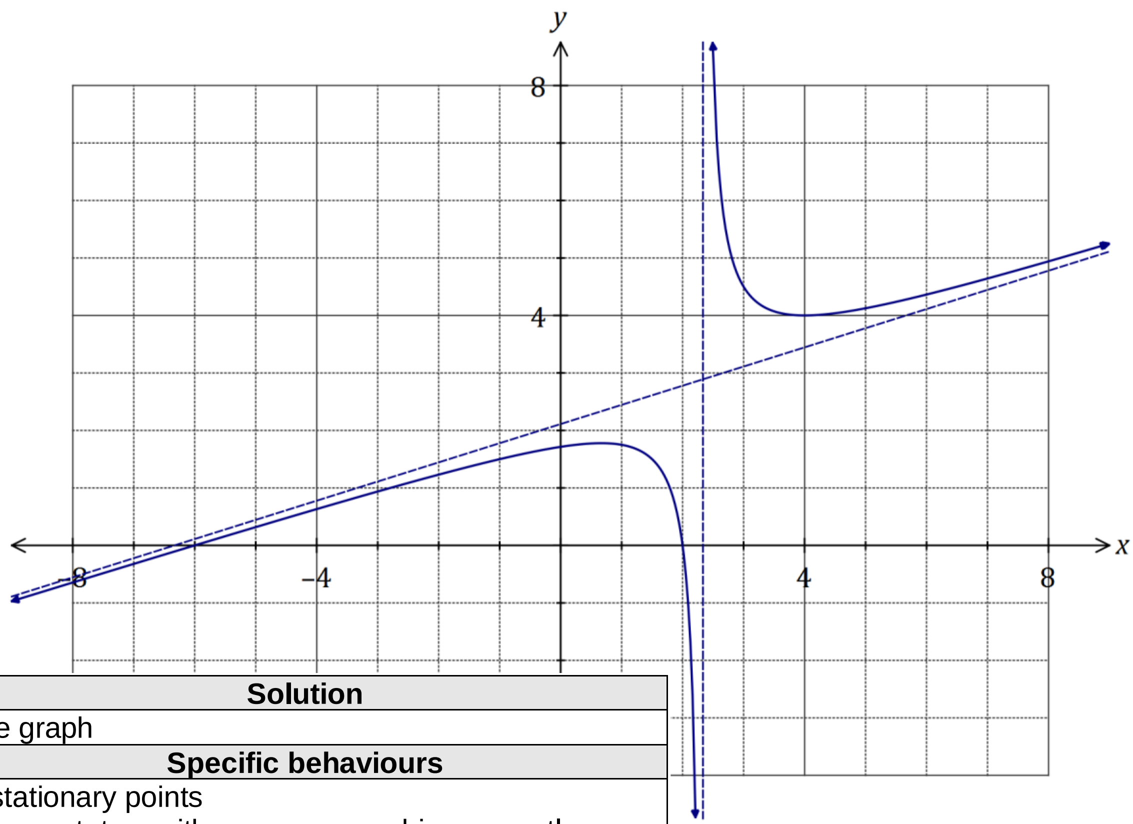
(a) Determine the exact coordinates of all stationary points of the graph of $y = f(x)$. (2 marks)

Solution
$f'(x) = 0$ when $(x-4)(3x-2) = 0 \Rightarrow x = 4, x = \frac{2}{3}$ At $(4, 4)$ and $\left(\frac{2}{3}, \frac{16}{9}\right)$
Specific behaviours
✓ first point ✓ second point

(b) Determine the equation(s) of the asymptote(s) of the graph $y = f(x)$. (3 marks)

Solution
Vertical asymptote: $x = \frac{7}{3}$ $f(x) = \frac{x}{3} + \frac{19}{9} + \frac{25}{9(3x-7)}$ Oblique asymptote: $y = \frac{x}{3} + \frac{19}{9}$
Specific behaviours
✓ vertical asymptote ✓ indicates equivalent form of f ✓ oblique asymptote

(c) Sketch the graph $y = f(x)$ on the axes below. (4 marks)

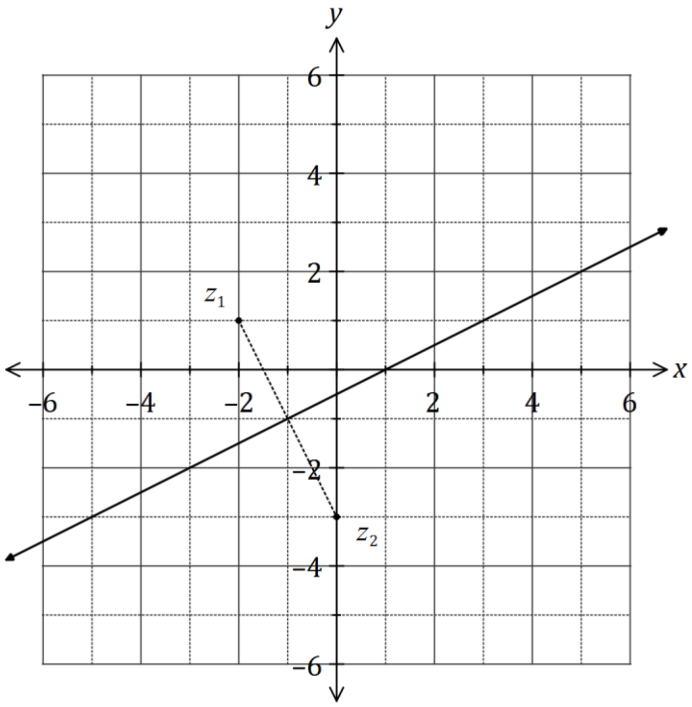


Solution
See graph
Specific behaviours
✓ stationary points ✓ asymptotes, with curve approaching correctly ✓ intercepts ✓ smooth curves

Question 13 **(11 marks)**

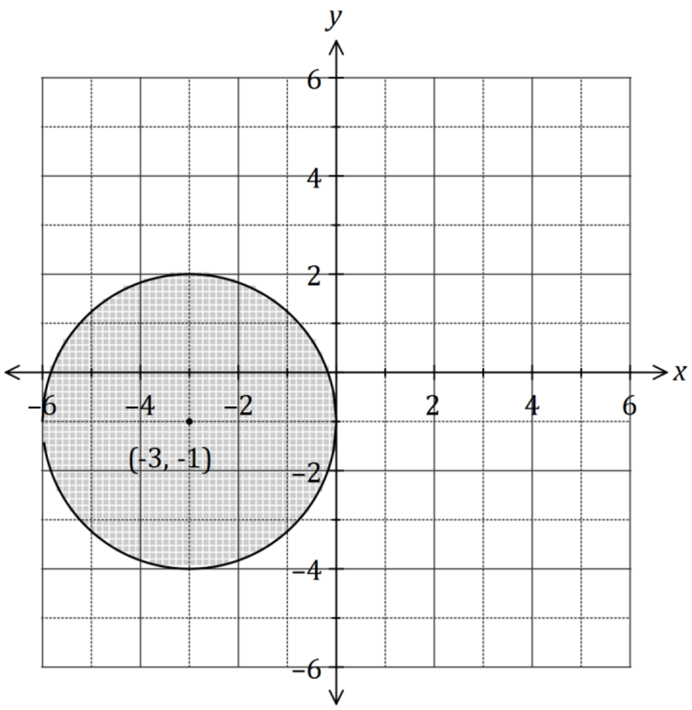
(a) On the Argand planes below, sketch the subsets of the complex plane determined by

(i) $|z+3i| = |z+2-i|$. (3 marks)



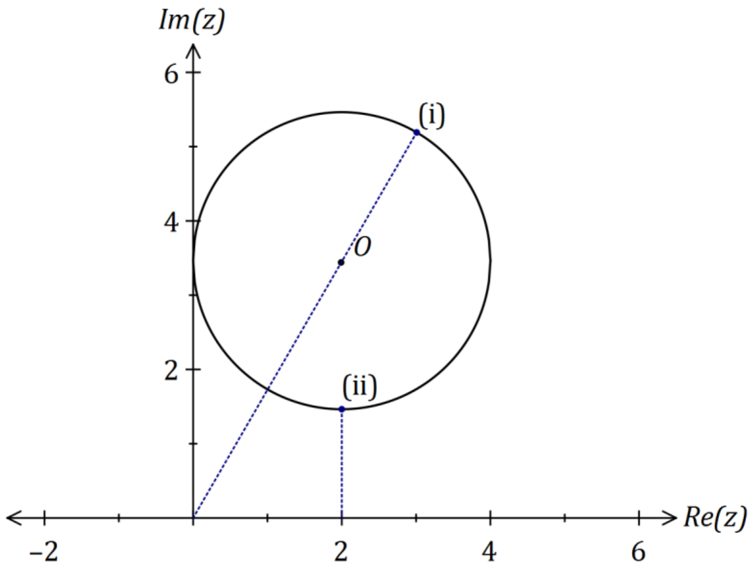
Solution
$\lvert z-(0-3i)\rvert \le \lvert z-(-2+i)\rvert$ Boundary is $y=\frac{1}{2}x-\frac{1}{2}$ See graph
Specific behaviours
✓ indicates use of (0, -3) and (-2, 1) ✓ indicates boundary is straight line ✓ indicates accurate boundary

(ii) $|z+3+i| \le 3$. (3 marks)



Solution
$\lvert z-(-3-i)\rvert \le 3$ See graph
Specific behaviours
✓ indicates circle, including boundary ✓ uses correct centre and radius ✓ shades inner region

(b) A subset of the complex plane, a circle with centre O, is shown below.



(i) Mark the position in the plane where $|z|$ is maximised. Label this point (i). (1 mark)

Solution
Maximum when z lies on circumference at greatest distance from origin.
Specific behaviours
✓ indicates location

(ii) Mark the position in the plane where $|z-2|$ is minimised. Label this point (ii). (1 mark)

Solution
Minimum when z lies on circumference at closest point to (2, 0).
Specific behaviours
✓ indicates location

(iii) If the subset shown is $\left|z-2-2\sqrt{3}i\right|=2$, determine the maximum and minimum values of $\arg z$. (3 marks)

Solution
Maximum: $\arg z=\frac{\pi}{2}$ Centre: $\arg(2+2\sqrt{3}i)=\frac{\pi}{3}$ Minimum: $\arg z=\frac{\pi}{2}-\frac{\pi}{3}=\frac{\pi}{6}$
Specific behaviours
✓ states maximum   ✓ indicates argument of centre   ✓ uses symmetry to determine minimum

Question 14 **(8 marks)**

The plane P has equation $\mathbf{r}\cdot\mathbf{n} = 11$, where $\mathbf{n} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and the point A has position vector $2\boldsymbol{i} + 5\mathbf{j} - 2\mathbf{k}$.

(a) Determine the Cartesian equation of plane Q that is parallel to P and passes through A. (2 marks)

Solution
$x - y + 2z = (2) - (5) + 2(-2)$ $x - y + 2z = -7$
Specific behaviours
✓ writes LHS of equation ✓ determines constant

(b) Determine the equation of the line L that passes through A and is perpendicular to P. (1 mark)

Solution
$\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$
Specific behaviours
✓ writes equation

(c) Determine the position vector of B, the point of intersection of line L with plane P. (3 marks)

Solution
$\begin{pmatrix} 2+\lambda \\ 5-\lambda \\ -2+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 11$ $2 + \lambda - 5 + \lambda - 4 + 4\lambda = 11 \Rightarrow \lambda = 3$ $\overrightarrow{OB} = \begin{pmatrix} 2+3 \\ 5-3 \\ -2+2(3) \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$
Specific behaviours
✓ substitutes line into plane ✓ solves for parameter ✓ states point of intersection

(d) Determine the exact distance between planes P and Q. (2 marks)

Solution
$\overrightarrow{AB} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 6 \end{pmatrix}$ $\lvert\overrightarrow{AB}\rvert = 3\sqrt{6}$
Specific behaviours
✓ determines $\overrightarrow{AB}$ ✓ states distance

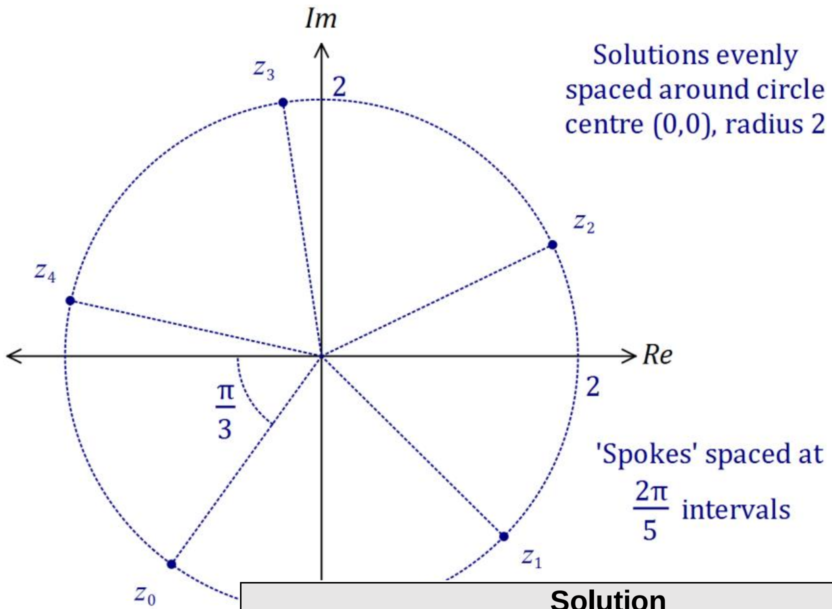
Question 15 **(8 marks)**

Consider the complex equation $z^5 = -16 + 16\sqrt{3}i$.

(a) Solve the equation, giving all solutions in the form $r \operatorname{cis} \theta$, $r > 0$, $-\pi \leq \theta \leq \pi$. (4 marks)

Solution
$z^5 = 32 \operatorname{cis}\left(\frac{2\pi}{3}\right)$ $z = 2 \operatorname{cis}\left(\frac{2\pi}{15} + \frac{2k\pi}{5}\right), k \in \mathbb{Z}$ $z_0 = 2 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$ $z_1 = 2 \operatorname{cis}\left(-\frac{4\pi}{15}\right)$ $z_2 = 2 \operatorname{cis}\left(\frac{2\pi}{15}\right)$ $z_3 = 2 \operatorname{cis}\left(\frac{8\pi}{15}\right)$ $z_4 = 2 \operatorname{cis}\left(\frac{14\pi}{15}\right)$
Specific behaviours
✓ expresses in polar form ✓ uses de Moivre's theorem to obtain general solution ✓ states one correct root ✓ states all roots

(b) Plot the solutions found in part (a) on the Argand diagram below, indicating all key features of the plot. (4 marks)



Solution
See diagram
Specific behaviours
✓ indicates solutions at distance 2 from origin ✓ indicates solutions evenly spaced at $\frac{2\pi}{5}$ intervals ✓ exact argument of one solution shown ✓ indicates approximate position of all five solutions

Question 16 **(12 marks)**

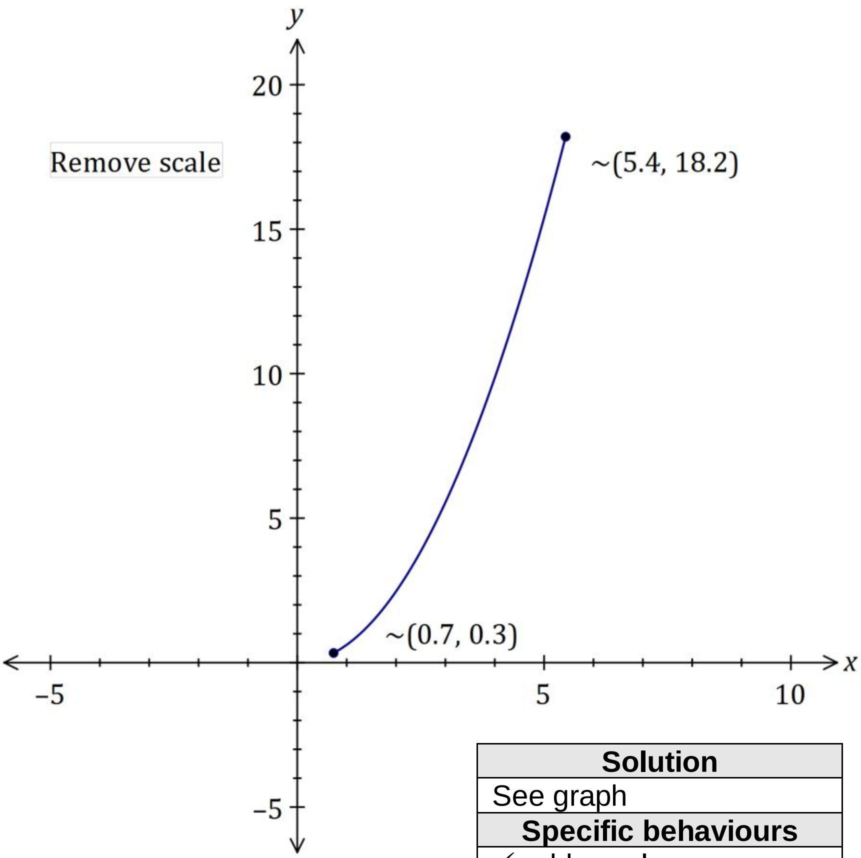
The position vector, in centimetres, of a particle at time t seconds is given below.

$$\mathbf{r}(t) = 2e^{t-1}\mathbf{i} + \frac{e^{2t}}{3}\mathbf{j}$$

(a) Show that the path of the particle can be expressed as a Cartesian equation in the form $y = ax^2$, and determine the value of a. (4 marks)

Solution
$x = 2e^{t-1} = \frac{2}{e}e^t \Rightarrow e^t = \frac{xe}{2}$ $y = \frac{e^{2t}}{3} = \frac{1}{3}(e^t)^2$ $y = \frac{1}{3}\left(\frac{xe}{2}\right)^2 = \frac{e^2}{12}x^2$ $a = \frac{e^2}{12}$
Specific behaviours
✓ expresses e^t in terms of x ✓ expresses y in terms of e^t ✓ substitutes and simplifies ✓ states value

(b) Sketch the path of the particle for $0 \le t \le 2$. (3 marks)



Solution
See graph
Specific behaviours
✓ adds scale ✓ endpoints ✓ parabolic shape

(c) Determine the speed of the particle when $t = 1$. (3 marks)

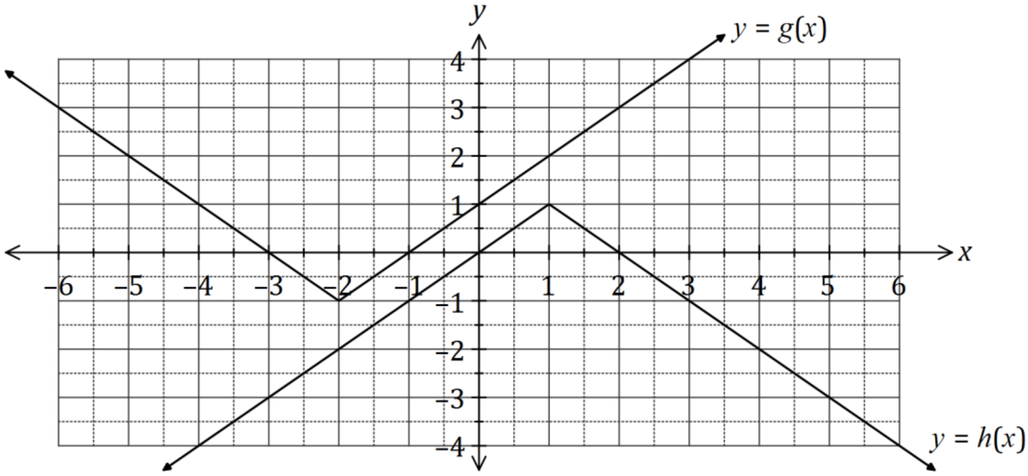
Solution
$\mathbf{v}(t) = 2e^{t-1}\mathbf{i} + \dfrac{2e^t}{3}\mathbf{j}$ $\mathbf{v}(1) = 2\mathbf{i} + \dfrac{2e^2}{3}\mathbf{j}$ $\lvert\mathbf{v}\rvert = \dfrac{2}{3}\sqrt{e^4 + 9} \approx 5.32 \text{ cm/s}$
Specific behaviours
✓ differentiates ✓ substitutes ✓ determines speed

(d) Write an expression in terms of t for the total distance travelled by the particle along its path between $t = 0$ and $t = 2$. Do **not** evaluate this expression. (2 marks)

Solution
$\lvert\mathbf{v}\rvert = \dfrac{2\sqrt{9e^{2(t-1)} + e^{4t}}}{3}$ Distance: $\displaystyle\int_0^2 \frac{2\sqrt{9e^{2(t-1)} + e^{4t}}}{3}\, dt$
Specific behaviours
✓ writes expression for the speed of the particle ✓ writes integral with correct bounds and wrt t

Question 17 **(9 marks)**

(a) The graphs of the functions g and h are shown below.



Determine the value(s) of k if

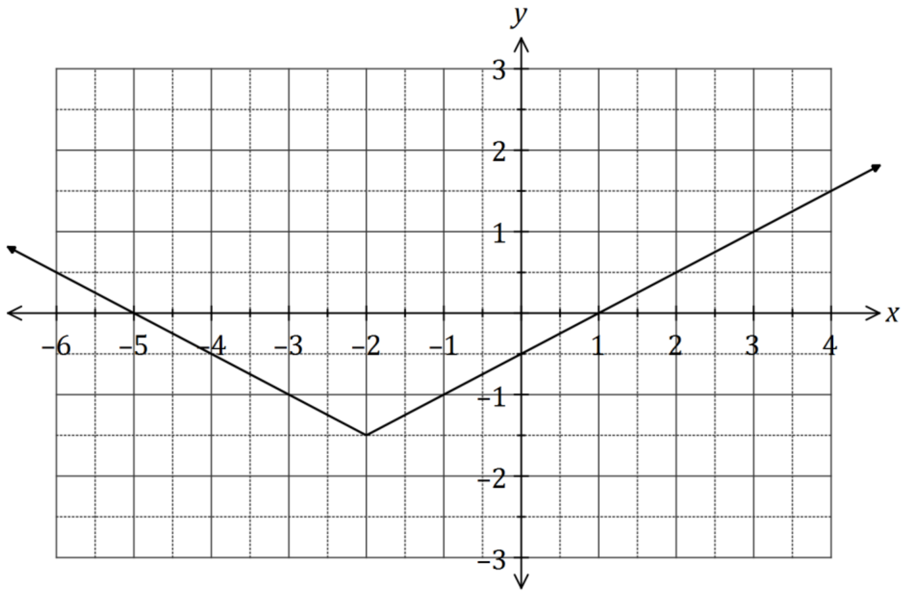
(i) $k = h \circ g(2)$. (1 mark)

Solution
$k = h(3) = -1$
Specific behaviours
✓ correct value

(ii) $g\big(h(k)\big) = 1$. (2 marks)

Solution
$g(x) = 1 \Rightarrow x = 0, -4$ $h(k) = 0 \Rightarrow k = 0, 2$ and $h(k) = -4 \Rightarrow k = -4, 6$ $k = -4, 0, 2, 6$
Specific behaviours
✓ indicates $h(k) = 0$ or -4 ✓ states all 4 values

(b) The graph of $f(x) = a|x - p| + q$ is shown below.



(i) Determine the value of the constants a, p and q. (3 marks)

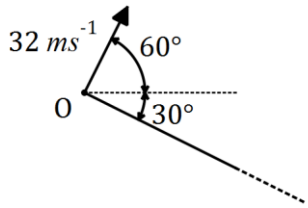
Solution
Gradient: $a = \frac{1}{2}$ Vertical translation: $q = -1.5$ Horizontal translation: $p = -2$
Specific behaviours
✓✓✓ each value

(ii) If the equation $|f(x)| = mx + c$ has an infinite number of solutions, determine the values of the positive constants m and c. (3 marks)

Solution
For infinite solutions, we require straight line to be a part of $\|f(x)\|$. For m and c to be positive, must be part of $\|f(x)\| \text{where} -5 \leq x \leq -2$ $m = \frac{1}{2}$ and $c = \frac{5}{2}$
Specific behaviours
✓ indicates must be one of four segments of $\|f(x)\|$ ✓ determines m ✓ determines c

Question 18 **(12 marks)**

A small body is projected upwards from the top of a hill with an initial velocity of 32 ms^{-1} at an angle of 60° to the horizontal. The hill slopes downwards at a constant angle of 30° to the horizontal. Let the origin O of a cartesian coordinate system be the top of the hill, with $\mathbf{i}$ a unit vector in the positive x direction and $\mathbf{j}$ a unit vector in the positive y direction. Displacement is measured in metres and time in seconds.



(a) Show that the initial velocity of the body is $16\mathbf{i} + 16\sqrt{3}\mathbf{j}$. (1 mark)

Solution
$\mathbf{v}(0) = 32\cos 60\,\mathbf{i} + 32\sin 60\,\mathbf{j}$ $= 16\mathbf{i} + 16\sqrt{3}\mathbf{j}$
Specific behaviours
✓ uses trig ratios

The acceleration of the body, t seconds after projection, is given by $\mathbf{a} = -0.2t\mathbf{i} + (0.2t - 10)\mathbf{j}$.

(b) Determine an expression for the position vector of the body after t seconds. (3 marks)

Solution
$\mathbf{v} = \left(16 - \frac{t^2}{10}\right)\mathbf{i} + \left(16\sqrt{3} + \frac{t^2}{10} - 10t\right)\mathbf{j}$ $\mathbf{r} = \left(16t - \frac{t^3}{30}\right)\mathbf{i} + \left(16\sqrt{3}t + \frac{t^3}{30} - 5t^2\right)\mathbf{j}$
Specific behaviours
✓ integrates acceleration ✓ uses initial velocity for constant ✓ integrates velocity

(c) Determine the time at which the body lands on the hillside. (3 marks)

Solution
$\frac{\mathbf{i}\text{-coeff}}{\mathbf{j}\text{-coeff}} = \tan(-30°)$ $16\sqrt{3}t + \frac{t^3}{30} - 5t^2 = \left(16t - \frac{t^3}{30}\right) \times \tan(-30°)$ $t = 7.551\text{ s}$
Specific behaviours
✓ uses ratio of coefficients and tangent of slope ✓ writes equation using position coefficients ✓ solves equation

(d) Calculate the distance of the body from O at the instant it lands. (2 marks)

Solution
$\mathbf{r}(7.551) = (106.46)\mathbf{i} + (-61.47)\mathbf{j}$ $\lvert\mathbf{r}\rvert = 122.9 \text{ m}$
Specific behaviours
✓ determines position ✓ calculates magnitude

(e) Determine the maximum vertical height attained by the particle above the hillside. (3 marks)

Solution
$h = 16\sqrt{3}t + \frac{t^3}{30} - 5t^2 + \tan(30)\, t$ $\frac{dh}{dt} = \frac{3t^2 - 300t + 490\sqrt{3}}{30}$ $\frac{dh}{dt} = 0 \Rightarrow t = 2.914 \text{ s}$ $h(2.914) = 40.8 \text{ m}$
Specific behaviours
✓ forms equation for height above hillside ✓ solves equation for maximum ✓ determines height

Question 19 **(6 marks)**

Determine, where possible, a unique solution for the following systems of equations. In each case, interpret the system of equations geometrically.

(a) $8x + y + z = 15$, $2x + y - z = 3$, and $x - y + 2z = 3$. (2 marks)

Solution
No unique solution, as infinite number of solutions exist. As planes clearly not parallel, then they represent three planes that intersect in a straight line. $(x = t, y = -5t + 9, z = 6 - 3t)$
Specific behaviours
✓ indicates infinite number of solutions ✓ indicates planes intersecting in a straight line

(b) $x + y - z = 0$, $x - y + 2z = 10$ and $3x - y + z = 16$. (2 marks)

Solution
Using CAS, $x = 4, y = -2, z = 2$ Three planes that intersect at the point $(4, -2, 2)$.
Specific behaviours
✓ solution ✓ interpretation

(c) $x + y = z + 2$, $x - y + z = 1$ and $x + z = y + 3$. (2 marks)

Solution
No solutions exist. Two parallel planes cut by the other plane. *(Last plane can be written $x - y + z = 3$ - parallel and non-intersecting with $x - y + z = 1$).*
Specific behaviours
✓ indicates no solutions ✓ indicates two parallel planes cut by third

End of questions

Additional working space

Question number: _________